ME 261: Numerical Analysis

Lecture-5: Root Finding

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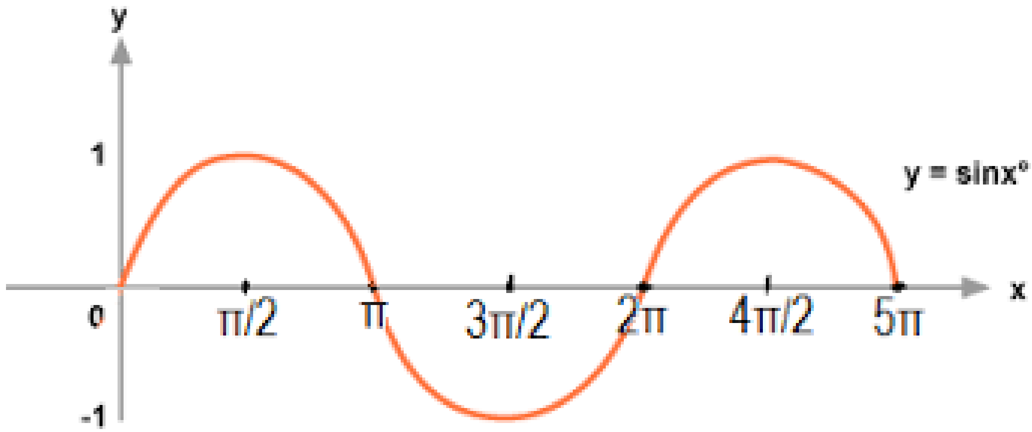
- **One of the most common problem encountered in engineering analysis is that given a function f (x), find the values of x for which f (x) = 0.**
- **The solution (values of x) are known as the roots of the equation f (x) = 0, or the zeroes of the function f (x).**

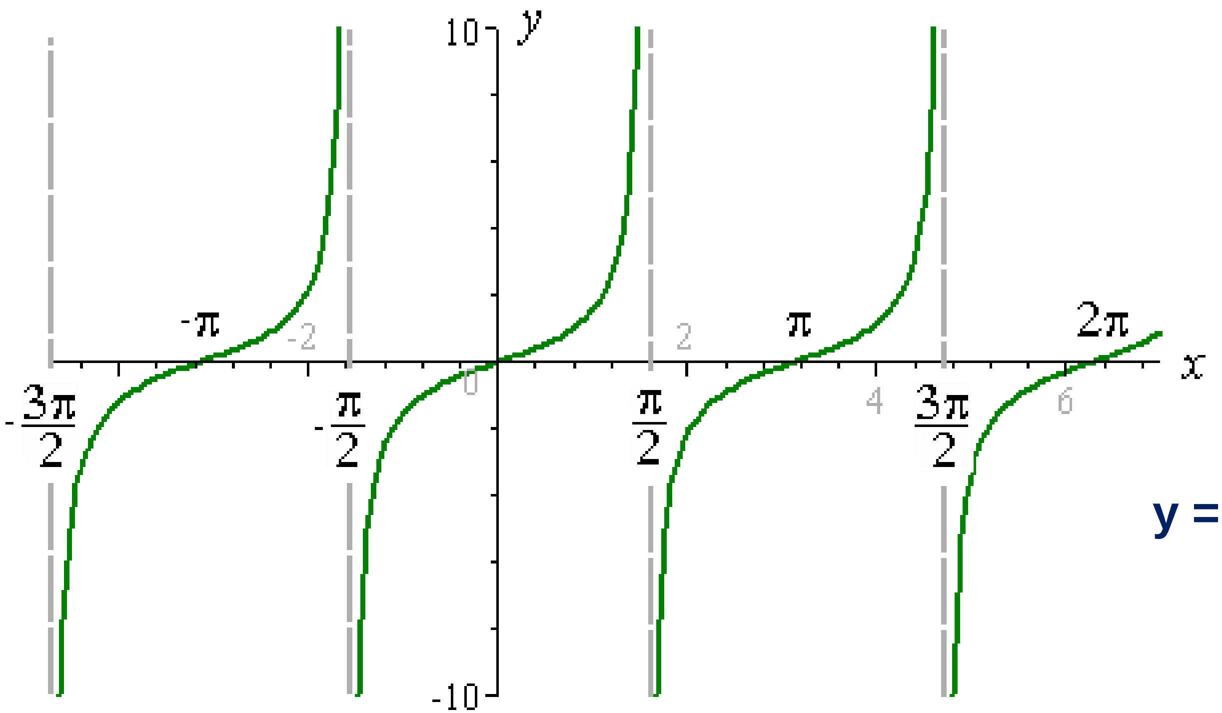
$$f(x) = ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$f(x) = e^{-x} - x$$

- The roots of equations **may be real or complex.**
- In general, an equation may have **any number of (real) roots, or no roots** at all. For example:
  - sin x – x = 0 has a single root, at x = 0,
  - tan x – x = 0 has infinite number of roots



y = sin x



y = tan x

- Equations may be of two types:
  - **Algebraic**
  - **Transcendal**

A function given by y = f(x) is **algebraic if it can be expressed** in the form:

$$f_n y^n + f_{n-1} y^{n-1} + \cdots + f_1 y + f_0 = 0$$

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where n = the order of the polynomial and the a's = constants.

$$f_6(x) = 5x^2 - x^3 + 7x^6$$

A transcendental function is one that is **nonalgebraic**. These include **trigonometric, exponential, logarithmic**, and other, less familiar, functions.

$$f(x) = \ln x^2 - 1$$

$$f(x) = e^{-0.2x} \sin(3x - 0.5)$$

Roots of equations

Bracketing methods

Root is to be located within an interval prescribed by **a lower and upper bound**.

Such methods are said to be **convergent** because they move closer to the truth as the computation progresses.

- **Bisection method**
- **False-position method (regula falsi)**

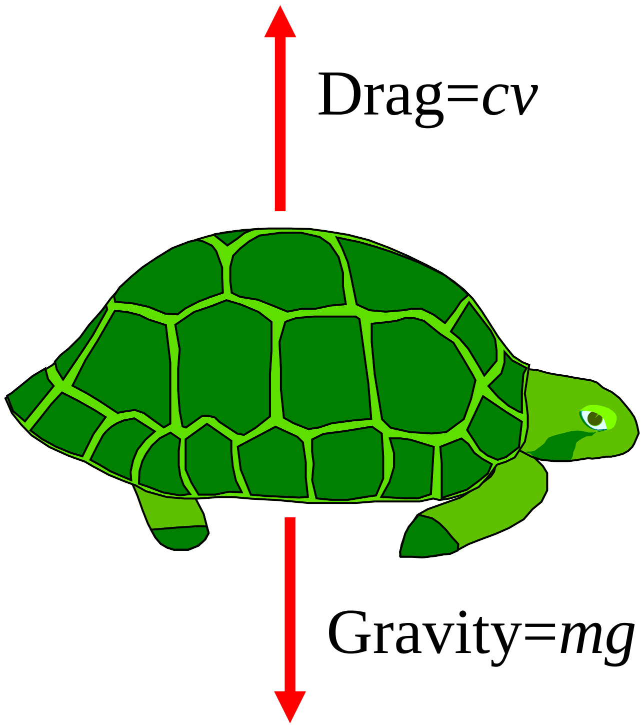
Open methods

Require only **a single initial (starting value) or two starting values** that do not necessarily bracket the root.

They sometimes **diverge or move away** from the true root as the computation progresses. However, when the open methods converge, they usually do so much **more quickly** than the bracketing methods.

- **Fixed-point iteration method**
- **Newton-Raphson method (Newton's method)**
- **Secant method**

Free Falling of a Turtle: A simple mathematical model



m= mass of the turtle,
v = velocity of the turtle
t = time
c = Drag Coefficient
g = gravitational acceleration

$$m\frac{dv}{dt} = mg - cv \qquad (1)$$

$$\frac{dv}{dt} = g - \frac{c}{m}v \qquad (2)$$

Eq. (2) is a **first order linear differential eqn.** If v=0 at t=0

$$v(t) = \frac{gm}{c}\left(1 - e^{-\frac{ct}{m}}\right) \qquad (3) ?????$$

Problem 1.

Given that the velocity of the turtle is governed by Eq. (3), **find the value of the drag coefficient, *c*,** such that a turtle of mass, ***m*=5 *kg***, can attain a prescribed velocity, ***v*=10 *m* /*s*,** at a set period of time, ***t*=9 *s***. Use ***g* = 9.81m/s²**.

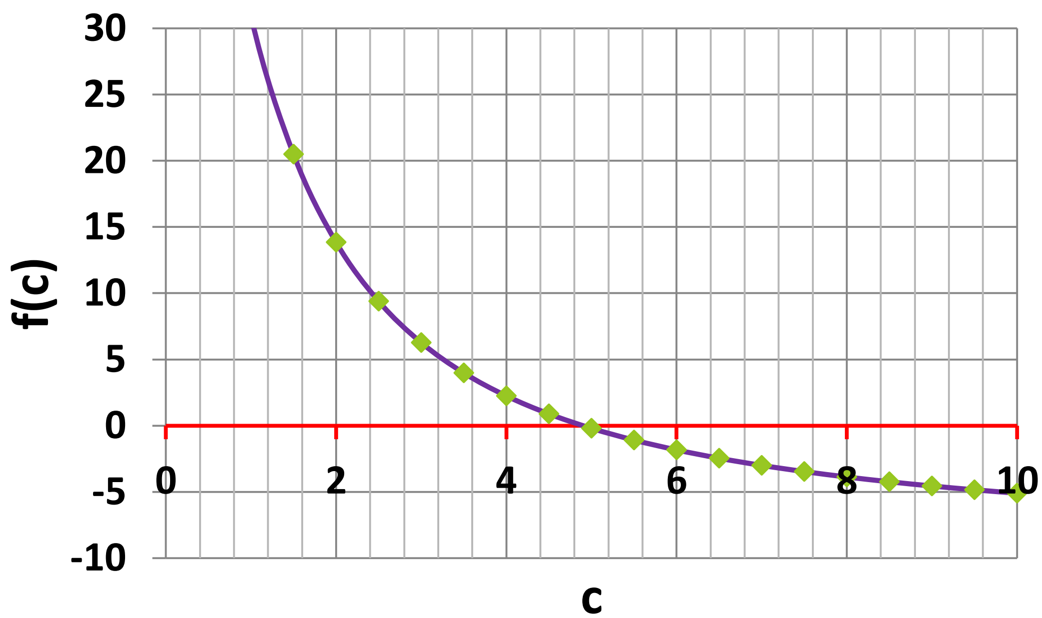
$$\frac{gm}{c}\left(1-e^{-\frac{ct}{m}}\right)-v(t)=0 \qquad (4)$$

$$f(c)=\frac{gm}{c}\left(1-e^{-\frac{ct}{m}}\right)-v(t)=0 \qquad (5)$$

$$f(c)=\frac{49.05}{c}\left(1-e^{-1.8c}\right)-10=0 \qquad (6)$$

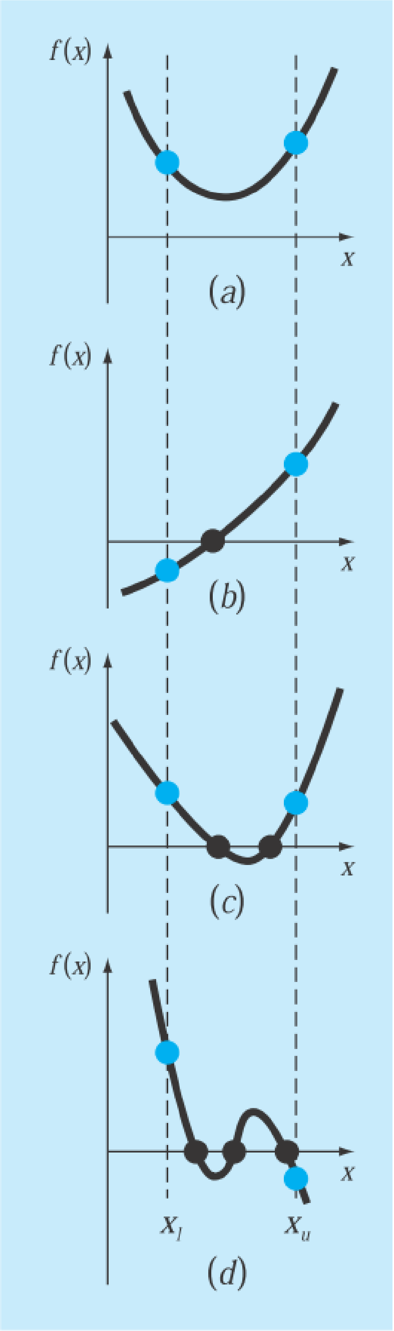
Graphical Method for Finding Root

c	f(c)
1	30.94209
1.5	20.50238
2	13.85489
2.5	9.402041
3	6.276154
3.5	3.988551
4	2.253345
4.5	0.896691
5	-0.19121
5.5	-1.08227
6	-1.82517
6.5	-2.45391
7	-2.99288
7.5	-3.46001
8	-3.86875
8.5	-4.22941
9	-4.55
9.5	-4.83684
10	-5.095

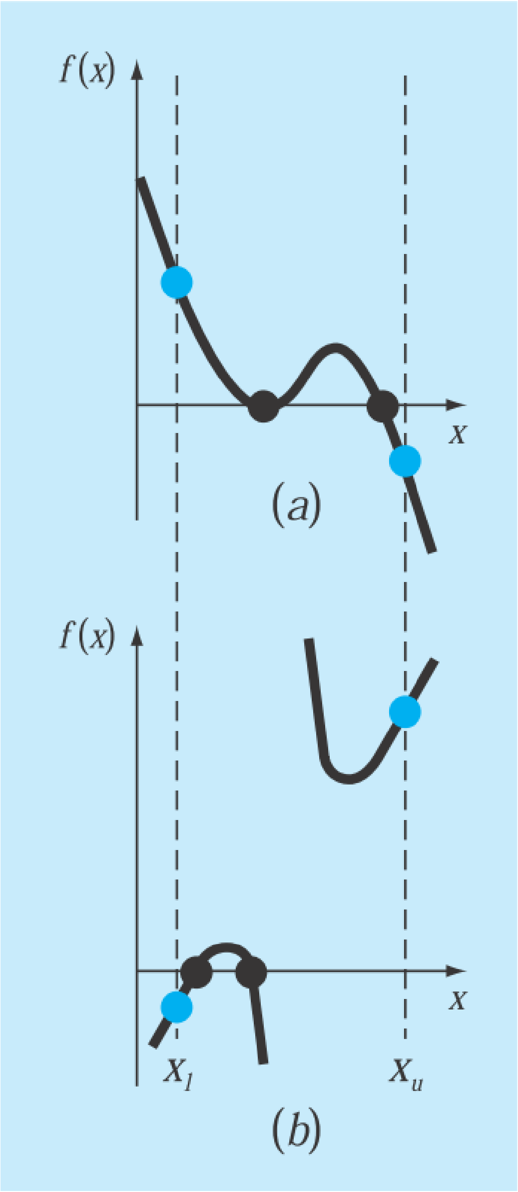


Approximate root~ 4.9

- Graphical techniques are of **limited practical importance** because they are not **precise**.
- However, graphical methods can be utilized to obtain **rough estimates of roots**. These estimates can be employed as **starting guesses** for other numerical methods discussed in the next.
- Graphical interpretations are important tools for understanding **the properties of the functions** and anticipating the **pitfalls of the numerical methods**.
- In general, if $f(x_l)$ and $f(x_u)$ have **opposite signs**, there are **an odd number** of roots in the interval.
- If $f(x_l)$ and $f(x_u)$ have the **same sign**, there are either **no roots or an even number of roots** between the values.



There are always some exceptions!!



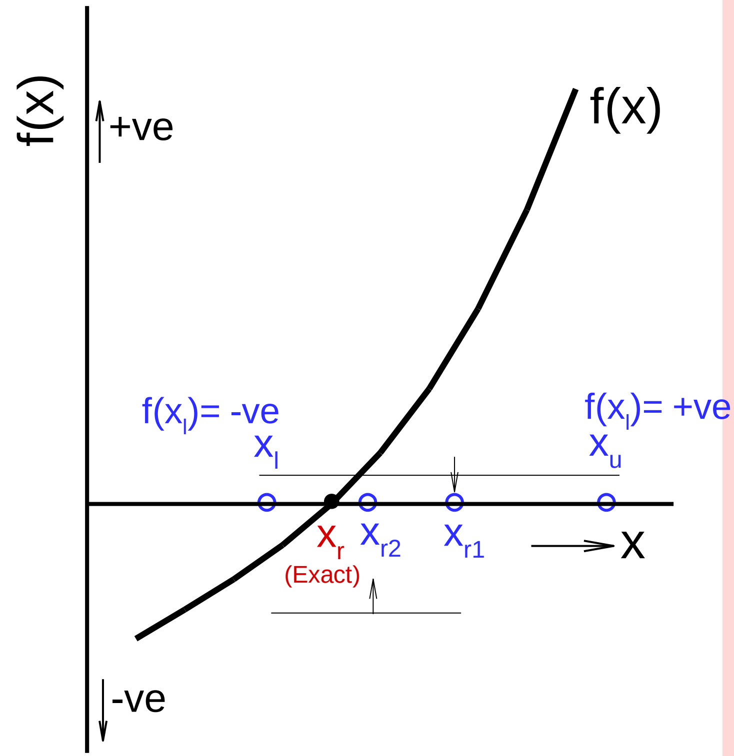
Bisection Method for Finding Root

If f(x) is **real and continuous** in the interval from x_l to x_u and **$f(x_l)$ and $f(x_u)$ have opposite sign** that is $f(x_l)\,f(x_u) < 0$

then there is **at least one root between x_l and x_u.**

In this method, the interval x_l and x_u is always divided in **half** to estimate the root as

$$x_r = \frac{x_l + x_u}{2}$$

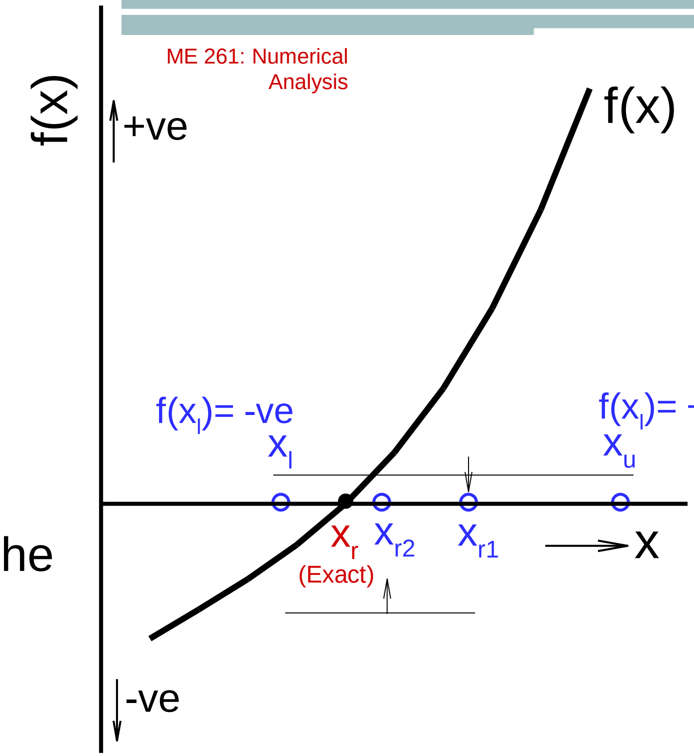


- Next sub-interval is identified based on the criteria of change of sign of functional values.
- The location of the root is then determined at the midpoint of the sub-interval within which the sign changes occurs.
- The process is repeated to obtain refined estimate.

$$\varepsilon_a \le \varepsilon_s \text{ (stopping criteria)}$$

where

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100\%$$



Stopping criteria is to be given in either of the two ways

(i) tolerance limit in %

(ii) no. of significant digits

Solve the above turtle motion problem by **Bisection method** within initial range of root from $x_l = 4$ **and** $x_u = 6$. Continue root finding approximate error falls less then **0.5%**.

$$f(c) = \frac{49.05}{c}\left(1 - e^{-1.8c}\right) - 10 = 0$$

Calculation Table

it	xl	xu	xr	f(xl)	f(xu)	f(xr)	Approx. error (%)
1	4	6	5	2.253345	-1.82517	-0.19121	_
2	4	5	4.5	2.253345	-0.19121	0.896691	10
3	4.5	5	4.75	0.896691	-0.19121	0.324317	5.555556
4	4.75	5	4.875	0.324317	-0.19121	0.059983	2.631579
5	4.875	5	4.9375	0.059983	-0.19121	-0.06719	1.282051
6	4.875	4.9375	4.90625	0.059983	-0.06719	-0.00401	0.632911
7	4.875	4.90625	4.890625	0.059983	-0.00401	0.027886	0.318471
8	4.890625	4.90625	4.898438	0.027886	-0.00401	0.011913	0.159744
9	4.898438	4.90625	4.902344	0.011912	-0.00401	0.003946	0.07975
10	4.902344	4.90625	4.904297	0.003946	-0.00401	-3.3E-05	0.039838
11	4.902344	4.904297	4.903321	0.003946	-3.3E-05	0.001956	0.019911

$$f(c) = \frac{49.05}{c}\left(1 - e^{-1.8c}\right) - 10 = 0$$

$$Root\ Bracket\ [4,6];\ Approx\ Error \leq 0.5\%;$$

$$True\ root: 4.90332$$

Error Analysis for Bisection Method

$$x_{u,1} - x_{l,1} = \frac{x_{u,0} - x_{l,0}}{2} = \frac{\Delta x_0}{2}$$

$$x_{u,2} - x_{l,2} = \frac{x_{u,1} - x_{l,1}}{2} = \frac{\Delta x_0}{2^2}$$

$$x_{u,3} - x_{l,3} = \frac{x_{u,2} - x_{l,2}}{2} = \frac{\Delta x_0}{2^3}$$

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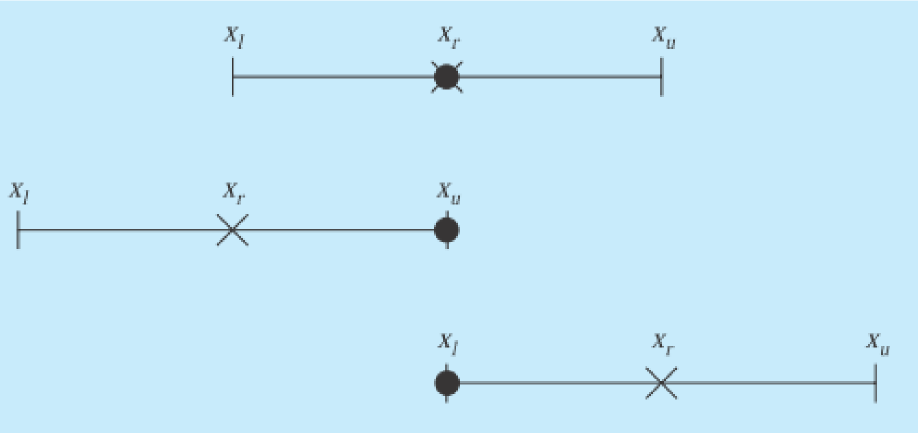
$$x_{u,n} - x_{l,n} = \frac{x_{u,n-1} - x_{l,n-1}}{2} = \frac{\Delta x_0}{2^n}$$



Maximum Possible Error bound in any iteration

$$\varepsilon_n \cancel{\leq} \left| x_{u,n} - x_{l,n} \right| \leq \frac{\Delta x_0}{2^n}$$

$$\varepsilon_n \cancel{\leq} \frac{\Delta x_0}{2^n}; 2^n \geq \frac{\Delta x_0}{\varepsilon_n}$$

The sign would be greater than or equal to

$$n \ln 2 \geq \ln \frac{\Delta x_0}{\varepsilon_n}; n \geq \frac{\ln(\Delta x_0/\varepsilon_n)}{\ln 2}; n \geq 1.44 \ln(\Delta x_0/\varepsilon_n)$$

Advantages:

- the finding of root is **guaranteed** since the interval always bracket the root.
- The **number of iterations** to achieve a **specific accuracy** is known in advance.

Disadvantages:

The approach is **slow to converge**.